Anticommutators and propagators of Moyal star-products for Dirac field on noncommutative spacetime

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Abstract

We study the Moyal anticommutators and their expectation values between vacuum states and non-vacuum states for Dirac fields on noncommutative spacetime. Then we construct the propagators of Moyal star-products for Dirac fields on noncommutative spacetime. We find that the propagators of Moyal star-products for Dirac fields are equal to the propagators of Dirac fields on ordinary commutative spacetime.

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Spacetime may have discrete and noncommutative structures under a very small microscopic scale. The concept of spacetime noncommutativity was first proposed by Snyder many years ago [1]. The purely mathematical development on noncommutative geometry was carried out by Connes [2]. To combine Heisenberg's uncertainty principle and Einstein's gravitational equations, Doplicher *et al.* proposed the uncertainty relations for the measurement of spacetime coordinates [3]. In recent years, spacetime noncommutativity was discovered again in superstring theories [4]. It has resulted a lot of researches on noncommutative field theories [5,6].

In noncommutative spacetime, we can regard the Moyal star-product as the basic product operation. Therefore we need to study the commutators and anticommutators and propagators of Moyal star-products for quantum fields on noncommutative spacetime. In Ref. [7] we studied the commutators and propagators of Moyal star-products for noncommutative scalar field theory. In this paper, we will study the anticommutators and propagators of Moyal star-products for noncommutative Dirac field.

In noncommutative spacetime, spacetime coordinates satisfy the commutation relation

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \ , \tag{1}$$

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where $\theta^{\mu\nu}$ is a constant real antisymmetric matrix that parameterizes the noncommutativity of the spacetime. There is a map between field theories on ordinary spacetime and on noncommutative spacetime. For field theories on noncommutative spacetime, they can be obtained through introducing the Moyal star-product, i.e., all of the products between field functions are replaced by the Moyal star-products. The Moyal star-product of two functions $f(x)$ and $g(x)$ is defined to be

$$\begin{aligned} f(x)\star g(x) &= e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\beta^\nu}} f(x+\alpha)g(x+\beta)|_{\alpha=\beta=0} \\ &= f(x)g(x)+\sum_{n=1}^{\infty}\left(\frac{i}{2}\right)^n\frac{1}{n!}\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}f(x)\partial_{\nu_1}\cdots\partial_{\nu_n}g(x)\ . \end{aligned} \tag{2}$$

The Moyal star-product of two functions of Eq. (2) is defined at the same spacetime point. We can generalize Eq. (2) to two functions on different spacetime points [6]:

$$\begin{aligned} f(x_1)\star g(x_2) &= e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\beta^\nu}} f(x_1+\alpha)g(x_2+\beta)|_{\alpha=\beta=0} \\ &= f(x_1)g(x_2)+\sum_{n=1}^{\infty}\left(\frac{i}{2}\right)^n\frac{1}{n!}\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}f(x_1)\partial_{\nu_1}\cdots\partial_{\nu_n}g(x_2)\ . \end{aligned} \tag{3}$$

Equation (3) can be established through generalize the commutation relation of spacetime coordinates at the same point to two different points:

$$[x_1^\mu, x_2^\nu] = i\theta^{\mu\nu}\ . \tag{4}$$

We can also expect to search for a grounds of argument for Eq. (4) from superstring theories.

To consider the free Dirac field on noncommutative spacetime, its Lagrangian is given by

$$\mathcal{L} = \overline{\psi}\star i\gamma^\mu\partial_\mu\psi - m\overline{\psi}\star\psi\ . \tag{5}$$

The Fourier expansions for the free Dirac fields are given by

$$\begin{aligned} \psi(\mathbf{x},t) &= \int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_{s=1,2}[b(p,s)u(p,s)e^{-ip\cdot x}+d^\dagger(p,s)v(p,s)e^{ip\cdot x}]\ , \\ \overline{\psi}(\mathbf{x},t) &= \int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_{s=1,2}[b^\dagger(p,s)\overline{u}(p,s)e^{ip\cdot x}+d(p,s)\overline{v}(p,s)e^{-ip\cdot x}]\ , \end{aligned} \tag{6}$$

where $E_p = p_0 = +\sqrt{|\mathbf{p}|^2+m^2}$. In Eq. (6), the spacetime coordinates are treated as noncommutative. They satisfy the commutation relations (1) and (4). The commutation relations for the creation and annihilation operators are still the same as in the commutative spacetime:

$$\begin{aligned} \{b(p,s), b^\dagger(p',s')\} &= \delta_{ss'}\delta^3(\mathbf{p}-\mathbf{p}')\ , \\ \{d(p,s), d^\dagger(p',s')\} &= \delta_{ss'}\delta^3(\mathbf{p}-\mathbf{p}')\ , \end{aligned}$$

$$
\begin{aligned}
\{b(p,s), b(p',s')\} &= \{d(p,s), d(p',s')\} = 0 \ , \\
\{b^\dagger(p,s), b^\dagger(p',s')\} &= \{d^\dagger(p,s), d^\dagger(p',s')\} = 0 \ , \\
\{b(p,s), d(p',s')\} &= \{b(p,s), d^\dagger(p',s')\} = 0 \ , \\
\{d(p,s), b(p',s')\} &= \{d(p,s), b^\dagger(p',s')\} = 0 \ .
\end{aligned} \tag{7}
$$

The spinors $u(p,s)$ and $v(p,s)$ satify

$$
\begin{aligned}
\sum_{s=1,2} u_\alpha(p,s)\overline{u}_\beta(p,s) &= \left(\frac{\not p + m}{2m}\right)_{\alpha\beta} \ , \\
\sum_{s=1,2} v_\alpha(p,s)\overline{v}_\beta(p,s) &= \left(\frac{\not p - m}{2m}\right)_{\alpha\beta} \ .
\end{aligned} \tag{8}
$$

We define the anticommutator of the Moyal star-product for Dirac field to be

$$
\{\psi_\alpha(x), \overline{\psi}_\beta(x')\}_\star = \psi_\alpha(x) \star \overline{\psi}_\beta(x') + \overline{\psi}_\beta(x') \star \psi_\alpha(x) \ . \tag{9}
$$

Equation (9) can be called the Moyal anticommutator for convenience. From the Fourier expansion of Eq. (6) for the free Dirac fields, we can calculate the Moyal anticommutator of Eq. (9). It is given by

$$
\begin{aligned}
\{\psi_\alpha(x), \overline{\psi}_\beta(x')\}_\star &= \int \frac{d^3p d^3p'}{(2\pi)^3} \frac{m}{\sqrt{E_p E_{p'}}} \Big\{ \sum_{s=1,2} [b(p,s)u_\alpha(p,s)e^{-ip\cdot x} + d^\dagger(p,s)v_\alpha(p,s)e^{ip\cdot x}], \\
&\qquad \sum_{s'=1,2} [b^\dagger(p',s')\overline{u}_\beta(p',s')e^{ip'\cdot x'} + d(p',s')\overline{v}_\beta(p',s')e^{-ip'\cdot x'}] \Big\}_\star \\
&= \int \frac{d^3p d^3p'}{(2\pi)^3} \frac{m}{\sqrt{E_p E_{p'}}} \sum_{s,s'} \Big[\Big\{ b(p,s)u_\alpha(p,s)e^{-ip\cdot x}, b^\dagger(p',s')\overline{u}_\beta(p',s')e^{ip'\cdot x'} \Big\}_\star \\
&\qquad + \Big\{ b(p,s)u_\alpha(p,s)e^{-ip\cdot x}, d(p',s')\overline{v}_\beta(p',s')e^{-ip'\cdot x'} \Big\}_\star \\
&\qquad + \Big\{ d^\dagger(p,s)v_\alpha(p,s)e^{ip\cdot x}, b^\dagger(p',s')\overline{u}_\beta(p',s')e^{ip'\cdot x'} \Big\}_\star \\
&\qquad + \Big\{ d^\dagger(p,s)v_\alpha(p,s)e^{ip\cdot x}, d(p',s')\overline{v}_\beta(p',s')e^{-ip'\cdot x'} \Big\}_\star \Big] \ .
\end{aligned} \tag{10}
$$

In Eq. (10) there exist two kinds of noncommutative structures, field operators and spacetime coordinates. Because the spacetime coordinates are noncommutative in Eq. (10), we cannot apply the anticommutation relations for the creation and annihilation operators of Eq. (7) directly to obtain a c-number result for the Moyal anticommutator.

In order to obtain a c-number result for the Moyal anticommutator, we can calculate its vacuum expectation value. We have

$$
\begin{aligned}
&\langle 0|\{\psi_\alpha(x), \overline{\psi}_\beta(x')\}_\star|0\rangle \\
= \ &\langle 0| \int \frac{d^3p d^3p'}{(2\pi)^3} \frac{m}{\sqrt{E_p E_{p'}}} \sum_{s,s'} \Big[\Big\{ b(p,s)u_\alpha(p,s)e^{-ip\cdot x}, b^\dagger(p',s')\overline{u}_\beta(p',s')e^{ip'\cdot x'} \Big\}_\star
\end{aligned}
$$

$$
\begin{aligned}
& \quad +\Big\{d^\dagger(p,s)v_\alpha(p,s)e^{ip\cdot x}, d(p',s')\overline{v}_\beta(p',s')e^{-ip'\cdot x'}\Big\}_\star\Big]|0\rangle \\
= & \int \frac{d^3p}{(2\pi)^3}\frac{m}{E_p}\sum_{s=1,2}[u_\alpha(p,s)\overline{u}_\beta(p,s)e^{-ip\cdot x}\star e^{ip\cdot x'} + v_\alpha(p,s)\overline{v}_\beta(p,s)e^{-ip\cdot x'}\star e^{ip\cdot x}] \\
= & \int \frac{d^3p}{(2\pi)^3 2E_p}[(\not p + m)_{\alpha\beta}e^{-ip\cdot x}\star e^{ip\cdot x'} + (\not p - m)_{\alpha\beta}e^{-ip\cdot x'}\star e^{ip\cdot x}] \\
= & \int \frac{d^3p}{(2\pi)^3 2E_p}\Big[(\not p + m)_{\alpha\beta}\exp(\frac{i}{2}p\times p)e^{-ip\cdot(x-x')} + (\not p - m)_{\alpha\beta}\exp(\frac{i}{2}p\times p)e^{ip\cdot(x-x')}\Big] \ ,
\end{aligned}
\tag{11}
$$

where $p\times q = p_\mu\theta^{\mu\nu}q_\nu$ and in the last equality we have applied the formula (3). Because $\theta^{\mu\nu}$ is antisymmetric, $p\times p = 0$, we obtain

$$
\begin{aligned}
& \langle 0|\{\psi_\alpha(x), \overline{\psi}_\beta(x')\}_\star|0\rangle \\
= & \int \frac{d^3p}{(2\pi)^3 2E_p}\Big[(\not p + m)_{\alpha\beta}e^{-ip\cdot(x-x')} + (\not p - m)_{\alpha\beta}e^{ip\cdot(x-x')}\Big] \\
= & (i\ \not\partial_x + m)_{\alpha\beta}\ i\Delta(x-x') \\
= & -iS_{\alpha\beta}(x-x') \ ,
\end{aligned}
\tag{12}
$$

where the singular function $\Delta(x-x')$ is defined to be [8,9]

$$
\Delta(x-y) = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{\omega_k}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\sin\omega_k(x_0-y_0) \ . \tag{13}
$$

So the result of Eq. (12) is equal to the anticommutator of Dirac field in ordinary commutative spacetime:

$$
\{\psi_\alpha(x), \overline{\psi}_\beta(x')\} = (i\ \not\partial_x + m)_{\alpha\beta}\ i\Delta(x-x') = -iS_{\alpha\beta}(x-x') \ . \tag{14}
$$

It is obvious to see that this equality relies on the antisymmetry of $\theta^{\mu\nu}$. We can also obtain

$$
\langle 0|\{\psi_\alpha(x), \psi_\beta(x')\}_\star|0\rangle = \langle 0|\{\overline{\psi}_\alpha(x), \overline{\psi}_\beta(x')\}_\star|0\rangle = 0 \ . \tag{15}
$$

When $(x-y)$ is a spacelike interval the singular function $\Delta(x-x')$ is zero. Therefore we can obtain the vacuum expectation value for the equal-time anticommutator to be [9]

$$
\langle 0|\{\psi_\alpha(\mathbf{x},t), \overline{\psi}_\beta(\mathbf{x}',t)\}_\star|0\rangle = -\gamma^0_{\alpha\beta}\partial_0\Delta(\mathbf{x}-\mathbf{x}', x^0 - x'^{\,0})|_{x^0=x'^{\,0}} = \gamma^0_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{x}') \ . \tag{16}
$$

We can also calculate the expectation values between non-vacuum states for the Moyal anticommutator (9). Let $|\Psi\rangle$ represent a normalized non-vacuum physical state for the system of Dirac field quanta in the occupation eigenstate:

$$
|\Psi\rangle = |N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_i}(s,s')\cdots,0\rangle \ . \tag{17}
$$

In Eq. (17) we use N_{p_i} to represent the occupation number for the momentum p_i, and use (s,s') to represent four kinds of the spinors $u(p,s)$ and $v(p,s)$. $N_{p_i}(s,s')$ can only take the

values 0 and 1. We suppose that the occupation numbers are nonzero only on some separate momentums p_i. For all other momentums, the occupation numbers are zero. We use 0 to represent that the occupation numbers are zero on all the other momentums and spins in Eq. (17). The state vector $|\Psi\rangle$ has the following properties [10]:

$$\begin{aligned}
&\langle N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_i}(s,s')\cdots|N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_i}(s,s')\cdots\rangle = 1 \ , \\
&\sum_{\mathrm{N_{p_1}(s,s')N_{p_2}(s,s')\cdots}} |N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_i}(s,s')\cdots\rangle\langle N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_i}(s,s')\cdots| = 1 \ ,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&a_{s,s'}(p_i)|N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_{i-1}}(s,s')0_{p_i}(s,s')N_{p_{i+1}}(s,s')\cdots\rangle = 0 \ , \\
&\quad a_{s,s'}(p_i)|N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_{i-1}}(s,s')1_{p_i}(s,s')N_{p_{i+1}}(s,s')\cdots\rangle \\
&= (-1)^{\sum\limits_{l=1}^{i-1} N_l(s,s')}|N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_{i-1}}(s,s')0_{p_i}(s,s')N_{p_{i+1}}(s,s')\cdots\rangle \ , \\
&a^{\dagger}_{s,s'}(p_i)|N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_{i-1}}(s,s')1_{p_i}(s,s')N_{p_{i+1}}(s,s')\cdots\rangle = 0 \ , \\
&\quad a^{\dagger}_{s,s'}(p_i)|N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_{i-1}}(s,s')0_{p_i}(s,s')N_{p_{i+1}}(s,s')\cdots\rangle \\
&= (-1)^{\sum\limits_{l=1}^{i-1} N_l(s,s')}|N_{p_1}(s,s')N_{p_2}(s,s')\cdots N_{p_{i-1}}(s,s')1_{p_i}(s,s')N_{p_{i+1}}(s,s')\cdots\rangle \ .
\end{aligned} \tag{19}$$

In Eq. (19), we use $a_{s,s'}$ to represent one kind of the annihilation operators $b(p,s)$ and $d(p,s)$, and use $a^{\dagger}_{s,s'}$ to represent one kind of the creation operators $b^{\dagger}(p,s)$ and $d^{\dagger}(p,s)$.

From Eq. (10) the expectation value between any non-vacuum state $|\Psi\rangle$ for the Moyal anticommutator is given by

$$\begin{aligned}
&\langle\Psi|\{\psi_\alpha(x),\overline{\psi}_\beta(x')\}_\star|\Psi\rangle \\
&= \langle\Psi|\int\frac{d^3pd^3p'}{(2\pi)^3}\frac{m}{\sqrt{E_pE_{p'}}}\sum_{s,s'}\Big[\Big\{b(p,s)u_\alpha(p,s)e^{-ip\cdot x},b^{\dagger}(p',s')\overline{u}_\beta(p',s')e^{ip'\cdot x'}\Big\}_\star \\
&\qquad +\Big\{d^{\dagger}(p,s)v_\alpha(p,s)e^{ip\cdot x},d(p',s')\overline{v}_\beta(p',s')e^{-ip'\cdot x'}\Big\}_\star\Big]|\Psi\rangle \\
&= \int\frac{d^3p}{(2\pi)^3}\frac{m}{E_p}\sum_{s=1,2}[u_\alpha(p,s)\overline{u}_\beta(p,s)e^{-ip\cdot x}\star e^{ip\cdot x'}+v_\alpha(p,s)\overline{v}_\beta(p,s)e^{-ip\cdot x'}\star e^{ip\cdot x}] \ .
\end{aligned} \tag{20}$$

To consider the different cases of the occupation numbers for the Dirac field quanta, in the second equality of Eq. (20), we need to consider the different orders for the Moyal star-products in fact. However because $e^{-ip\cdot x}\star e^{ip\cdot x'} = e^{ip\cdot x'}\star e^{-ip\cdot x}$ and $e^{-ip\cdot x'}\star e^{ip\cdot x} = e^{ip\cdot x}\star e^{-ip\cdot x'}$,

we can neglect these difference. Therefore we obtain

$$\begin{aligned}
& \langle\Psi|\{\psi_\alpha(x),\overline{\psi}_\beta(x')\}_\star|\Psi\rangle \\
= & \int\frac{d^3p}{(2\pi)^3 2E_p}[(\not p+m)_{\alpha\beta}e^{-ip\cdot x}\star e^{ip\cdot x'}+(\not p-m)_{\alpha\beta}e^{-ip\cdot x'}\star e^{ip\cdot x}] \\
= & \int\frac{d^3p}{(2\pi)^3 2E_p}\left[(\not p+m)_{\alpha\beta}e^{-ip\cdot(x-x')}+(\not p-m)_{\alpha\beta}e^{ip\cdot(x-x')}\right] \\
= & -iS_{\alpha\beta}(x-x')\ .
\end{aligned} \tag{21}$$

We can also obtain

$$\langle\Psi|\{\psi_\alpha(x),\psi_\beta(x')\}_\star|\Psi\rangle=\langle\Psi|\{\overline{\psi}_\alpha(x),\overline{\psi}_\beta(x')\}_\star|\Psi\rangle=0\ , \tag{22}$$

$$\langle\Psi|\{\psi_\alpha(\mathbf{x},t),\overline{\psi}_\beta(\mathbf{x}',t)\}_\star|\Psi\rangle=\gamma^0_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{x}')\ . \tag{23}$$

Thus the results of Eqs. (21) to (23) are equal to that of Eqs. (12), (15), and (16) of the vacuum state expectation values. We can see that the properties of the anticommutation relations for the creation and annihilation operators of Eq. (7) are still reflected in the above evaluations for the non-vacuum state expectation values of the Moyal anticommutators. Although the Lorentz invariant singular function $S(x-x')$ is zero for a spacelike interval of the two spacetime coordinates, we cannot deduce that Dirac fields on noncommutative spacetime satisfy the microscopic causality principle from Eqs. (12) and (21) as that for the scalar field case [7], for the reason that the physical observables for Dirac fields are not $\psi(x)$ and $\overline{\psi}(x)$ directly, they are some bilinear forms constructed from $\psi(x)$ and $\overline{\psi}(x)$. For the microscopic causality problem of Dirac fields on noncommutative spacetime, we will discuss it in a following paper.

The same as quantum fields on ordinary commutative spacetime [8,9], we can decompose the Fourier expansion of the free Dirac field into positive frequency part and negative frequency part:

$$\psi(x)=\psi^+(x)+\psi^-(x)\ , \tag{24}$$

where

$$\begin{aligned}
\psi^+(x) &= \int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_{s=1,2}b(p,s)u(p,s)e^{-ip\cdot x}\ , \\
\psi^-(x) &= \int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_{s=1,2}d^\dagger(p,s)v(p,s)e^{ip\cdot x}\ .
\end{aligned} \tag{25}$$

For the conjugate field we have

$$\overline{\psi}(x)=\overline{\psi}^+(x)+\overline{\psi}^-(x)\ , \tag{26}$$

$$\overline{\psi}^+(x)=\int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_{s=1,2}d(p,s)\overline{v}(p,s)e^{-ip\cdot x}\ ,$$

$$\overline{\psi}^-(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} \sum_{s=1,2} b^\dagger(p,s)\overline{u}(p,s)e^{ip\cdot x} \ . \tag{27}$$

According to Eq. (11), we can decompose the vacuum expectation value of the Moyal anticommutator into two parts:

$$\langle 0|\{\psi_\alpha(x), \overline{\psi}_\beta(x')\}_\star|0\rangle = \langle 0|\{\psi^+_\alpha(x), \overline{\psi}^-_\beta(x')\}_\star|0\rangle + \langle 0|\{\psi^-_\alpha(x), \overline{\psi}^+_\beta(x')\}_\star|0\rangle \ , \tag{28}$$

where

$$\begin{aligned}
& \langle 0|\{\psi^+_\alpha(x), \overline{\psi}^-_\beta(x')\}_\star|0\rangle \\
= \ & \langle 0| \int \frac{d^3p}{(2\pi)^3} \frac{m}{E_p} \sum_{s=1,2} \left\{ b(p,s)u_\alpha(p,s)e^{-ip\cdot x}, b^\dagger(p,s)\overline{u}_\beta(p,s)e^{ip\cdot x'} \right\}_\star |0\rangle \\
= \ & \int \frac{d^3p}{(2\pi)^3} \frac{m}{E_p} \sum_{s=1,2} u_\alpha(p,s)\overline{u}_\beta(p,s) e^{-ip\cdot x} \star e^{ip\cdot x'} \\
= \ & \int \frac{d^3p}{(2\pi)^3 2E_p} (\not{p} + m)_{\alpha\beta} e^{-ip\cdot(x-x')} \\
= \ & (i \not{\partial}_x + m)_{\alpha\beta} \ i\Delta^+(x-x') = -iS^+_{\alpha\beta}(x-x') \ ,
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
& \langle 0|\{\psi^-_\alpha(x), \overline{\psi}^+_\beta(x')\}_\star|0\rangle \\
= \ & \langle 0| \int \frac{d^3p}{(2\pi)^3} \frac{m}{E_p} \sum_{s=1,2} \left\{ d^\dagger(p,s)v_\alpha(p,s)e^{ip\cdot x}, d(p,s)\overline{v}_\beta(p,s)e^{-ip\cdot x'} \right\}_\star |0\rangle \\
= \ & \int \frac{d^3p}{(2\pi)^3} \frac{m}{E_p} \sum_{s=1,2} v_\alpha(p,s)\overline{v}_\beta(p,s) e^{-ip\cdot x'} \star e^{ip\cdot x} \\
= \ & \int \frac{d^3p}{(2\pi)^3 2E_p} (\not{p} - m)_{\alpha\beta} e^{ip\cdot(x-x')} = (i \not{\partial}_x + m)_{\alpha\beta} \ i\Delta^-(x-x') \\
= \ & -iS^-_{\alpha\beta}(x-x') = iS^+_{\alpha\beta}(x'-x) \ ,
\end{aligned} \tag{30}$$

because $\Delta^-(x-x') = -\Delta^+(x'-x)$. Thus we have

$$\langle 0|\{\psi_\alpha(x), \overline{\psi}_\beta(x')\}_\star|0\rangle = -i[S^+_{\alpha\beta}(x-x') + S^-_{\alpha\beta}(x-x')] \ , \tag{31}$$

and we have defined

$$S_{\alpha\beta}(x-x') = S^+_{\alpha\beta}(x-x') + S^-_{\alpha\beta}(x-x') \ . \tag{32}$$

We can also see that if we replace the vacuum state by any non-vacuum state in the above formulas the results will not change. The above results are also equal to the corresponding anticummutators of Dirac fields in ordinary commutative spacetime.

For the anticommutators of Eqs. (29) and (30), we can rewrite them furthermore:

$$\begin{aligned} & \langle 0|\{\psi^+_\alpha(x), \overline{\psi}^-_\beta(x')\}_\star|0\rangle = \langle 0|\psi^+_\alpha(x)\star\overline{\psi}^-_\beta(x')|0\rangle \\ = \ & \langle 0|\psi_\alpha(x)\star\overline{\psi}_\beta(x')|0\rangle = -iS^+_{\alpha\beta}(x-x') \ , \end{aligned} \tag{33}$$

$$\begin{aligned} & \langle 0|\{\psi^-_\alpha(x), \overline{\psi}^+_\beta(x')\}_\star|0\rangle = \langle 0|\overline{\psi}^+_\beta(x')\star\psi^-_\alpha(x)|0\rangle \\ = \ & \langle 0|\overline{\psi}_\beta(x')\star\psi_\alpha(x)|0\rangle = -iS^-_{\alpha\beta}(x-x') \ . \end{aligned} \tag{34}$$

We define the time-ordered Moyal star-product of two Dirac field operators to be

$$T\psi_\alpha(x)\star\overline{\psi}_\beta(x') = \theta(t-t')\psi_\alpha(x)\star\overline{\psi}_\beta(x') - \theta(t'-t)\overline{\psi}_\beta(x')\star\psi_\alpha(x) \ , \tag{35}$$

where $\theta(t-t')$ is the unit step function. We can calculate the vacuum expectation value of Eq. (35):

$$\langle 0|T\psi_\alpha(x)\star\overline{\psi}_\beta(x')|0\rangle = \theta(t-t')\langle 0|\psi_\alpha(x)\star\overline{\psi}_\beta(x')|0\rangle - \theta(t'-t)\langle 0|\overline{\psi}_\beta(x')\star\psi_\alpha(x)|0\rangle \ . \tag{36}$$

Equation (36) is just the Feynman propagator of Moyal star-product for Dirac field. We can call it the Feynman Moyal propagator for convenience. To introduce the singular function $S_F(x)$, we can write the Feynman Moyal propagator (36) as

$$\langle 0|T\psi_\alpha(x)\star\overline{\psi}_\beta(x')|0\rangle = iS_F(x-x')_{\alpha\beta} \ . \tag{37}$$

From Eqs. (33), (34), and (36), we have

$$S_F(x-x')_{\alpha\beta} = -[\theta(t-t')S^+_{\alpha\beta}(x-x') - \theta(t'-t)S^-_{\alpha\beta}(x-x')] \ , \tag{38}$$

where the momentum integral representation for the singular function $S_F(x-x')$ is given by

$$S_F(x-x') = \int \frac{d^4p}{(2\pi)^4}\frac{\not{p}+m}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-x')} \ . \tag{39}$$

From the above results, we can see that the Feynman Moyal propagator of Dirac field on noncommutative spacetime is just equal to the Feynman propagator of Dirac field on ordinary commutative spacetime. However it is necessary to point out that in Eq. (35) for the definition of the time-ordered Moyal star-product of two Dirac fields, we have made a simplified manipulation. This is because the Moyal star-products are not invariant generally under the exchange of the orders of two functions, for the second term of the right hand side of Eq. (35) we need to consider this fact. In the Fourier integral representation, it can be seen clearly that the second term of the right hand side of Eq. (35) will have an additional phase factor $e^{ip\times p'}$ in contrast to the first term of the right hand side of Eq. (35) due to the exchange of the order of $\psi_\alpha(x)$ and $\overline{\psi}_\beta(x')$ for the Moyal star-product. However in Eq. (36) when we calculate the vacuum expectation value for Eq. (35), we can see that the non-zero

contribution comes from the $p = p'$ part inside the integral (cf. Eqs. (29) and (30)). This will make the phase factor to be $e^{ip\times p}$, which is 1 due to the antisymmetry of $\theta^{\mu\nu}$. Thus in the right hand side of Eq. (35), we can omit this effect in the exchange of the order of two Dirac fields for their Moyal star-product equivalently.

Just like that in ordinary commutative spacetime, the physical meaning of the Feynman Moyal propagator (36) can also be explained as the vacuum to vacuum transition amplitude for Dirac fields on noncommutative spacetime. The reason why we would like to construct the Feynman propagators of Moyal star-products for quantum fields on noncommutative spacetime is that: for noncommutative field theories we can establish their S-matrix where the products between field operators in $\mathcal{H}_{int}$ are Moyal star-products. From the Wick's theorem expansion for the time ordered products of field operators, there will occur the Feynman Moyal propagators. Therefore it is necessary to study Feynman propagators of Moyal star-products for quantum fields on noncommutative spacetime.

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